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LIE'S VIEWS ON SEVERAL IMPORTANT POINTS IN MODERN MATHEMATICS.

By G. A. MILLER, Ph. D., Göttingen, Germany.

It is generally admitted that America has contributed comparatively little towards the advancement of the science of mathematics. During the last twenty years there has been a rapidly increasing progress in this direction. Several European countries have also moved forward at a rapid rate during this period, so that our relative position is not improving as rapidly as might be desired.

The standard of general scholarship required for the higher degrees at our better institutions is comparatively high but the number of important discoveries does not yet correspond to this standard. In fact, the two are not apt to advance very far together, for the field of mathematics is so extensive that most are compelled to choose between a superficial acquaintance with the whole range of mathematical research and an exhaustive knowledge of only a few subjects.

In view of these facts it is natural that there should be many who strive to lead American mathematical talent to those newer regions which seem to offer the most fruitful fields of investigation. While there is a great difference of opinion with respect to these regions yet the most successful investigators are in the best possible position to judge in regard to them.

The view expressed by Klein during last year, in his address on *Arithmetizing Mathematics*, that Lie in Leipzig, Germany, and Poincaré in Paris, France, are the two most active mathematical investigators of the present day, is quite generally held. The following translation of a part of the introductory remarks of an article* published during last year by the former of these may therefore be

*Berichte der Koenigl. Saechs. Gesellschaft, 1895.

of considerable interest, as it contains the views of the author in regard to several important points in mathematics, especially in regard to the most important newer regions.

"In this century the concepts known as substitution and substitution group, transformation and transformation group, operation and operation group, invariant, differential invariant, and differential parameter, appear continually more clearly as the most important concepts of mathematics. While the curve as the representation of a function of a single variable has been the most important object of mathematical investigation for nearly two centuries from Descartes, while on the other hand, the concept of transformation first appeared in this century as an expedient in the study of curves and surfaces, there has gradually developed in the last decades a general theory of transformations whose elements are presented by the transformation itself while the series of transformations, in particular the transformation groups, constitute the object.

The general theory of transformations is a branch of analysis in the sense that it can be developed by purely analytic methods. It has however the material geometrical property that its operations are not only conceivable but directly intuitive to a large extent.

If we consider that the difference between the analytic and the synthetic methods exists in the fact that the synthesist reasons with concepts while the analyst operates with symbols, according to fixed rules, we may see an important property of the theory of transformations in this that its theorems can be developed in an elegant analytic as well as in a perspicuous even intuitively clear manner. It is due to this fact that the theory of transformations is considerably simpler than the theory of substitutions.

It should be added that different branches of mathematics have contributed to the development of the theory of transformations and that many parts of mathematics have already been considerably advanced by means of this theory.

The theory of differential equations is the most important branch of mathematics. Each department of physics presents problems which depend upon the integration of differential equations. In general, the theory of differential equations involves the road towards the explanation of all natural phenomena which require time. While this theory has an infinite practical value it has also a corresponding theoretic importance since it leads in a rational manner to the study of new important functions and classes of functions."

Gottingen, Germany, October 26, 1896.

NUMBER, COUNTING, MEASUREMENT.

By GEORGE BRUCE HALSTED, M. A. (Princeton), Ph. D. (Johns Hopkins), Professor of Mathematics in the University of Texas, Austin, Texas.

Counting is essentially prior to measuring, but also the primary number concept is essentially prior to counting and necessary to explain the meaning, cause and aim of counting. It is here maintained that integral number had not a metric origin, nor was metric in its original purpose ; that integral number did not involve the idea of ratio, that in fact it was enormously simpler than that very delicate concept, *ratio*. Number is primarily a quality of an artificial individual. The stress laid upon it, the importance attached to this quality comes first from the advantage of being able to identify one of these artificial individuals. By artificial is meant "of human make." The characteristic of these artificial individuals is that each, though made an individual, is conceived as consisting of other individuals.

The primitive function of number is to serve the purposes of identification. But again, counting, which consists in associating with each primitive individual in an artificial individual a distinct primitive individual in a familiar artificial individual, is thus itself essentially the identification, by a one-to-one correspondence, of an unfamiliar with a familiar thing. Thus primitive counting decides which of the familiar groups of fingers is to have its numeric quality attached to the unfamiliar group counted.

This primitive use of number in defining by identification is illustrated by an ordinary pack of playing cards, where the identification of King, Queen, and Knave is not more clearly qualitative and opposed to every mode of measurement than is the identification of ace, deuce, and tray ; and indeed that the King outvalues the Knave has more to do with measurement than the fact that the ace outvalues the tray.

Counting implies first a known series of groups, mental wholes each made up of distinct wholes ; secondly an unfamiliar mental whole ; thirdly the identification of the unfamiliar group by its one-to-one correspondence with a familiar group of the known series.

Absolutely no idea of a unit, of measurement, of amount, of value or even of equality is necessarily involved or indeed ordinarily used. One counts when one wishes to find out whether the same group of horses has been driven back at night that were taken out in the morning ; where counting is a process of identification which it would seem intentionally humorous or comical to try to connect fundamentally with any idea of a unit of reference or of some *value* to be ascertained, or of the setting off of a horse as a sample unit of value and then equating the total value to the number of such units. Such an *argumentum in circulo* may perhaps be funny, but it is neither fact nor mathematics. Mathematics afterwards defines numerical equality by means of one-to-one correspondence,

which is absolutely distinct and away apart from the idea of ratio. We may say with perfect certainty that there is no implicit presence of the ratio idea in primitive number.

From the contemplation of the primitive individual in relation to the artificial individual spring the related ideas "one" and "many." An individual thought of in contrast to "a many" as not-many gives the idea of "a one." A many composed of "a one" and another "one" is characterized as "two." A many composed of "a one" and the special many "a two" is characterized as "three." And so on ; at first absolutely without counting, in fact before the invention of that patent process of identification now called counting. For a considerable period of its early life every child uses a number system consisting of only three terms, *one*, *two*, *many*, and no counting. As datum may be taken a psychical continuum, and distinctness may be found the outcome of a process of differentiation ; but what may be spoken of as the physically originated primitive individuals, however complete in their distinctness, have no numeric suggestion or quality. The intuitive but creative apperception and synthesis of a manifold must precede its conscious analysis which alone gives number. It is only to conceptual unities that the numeric quality pertains. Such conceptual unities are of human make and in a sense are not in nature, while on the other hand, though the world we consciously perceive is out and out a mental phenomenon, yet the primitive individuals, distinct things, while forming part of the artificial unities, exist in another way, in that they are subsisting somehow in nature as well as in conscious perception.

In reference to these fundamental matters some strange blunders have been made of late by eminent philosophers and teachers, not mathematicians.

The number-picture of a group is a selective photograph of the group, which takes or represents only one quality of the group, but takes that all at once.

This picture process only applies primarily to those particular artificial wholes which may be called discrete aggregates. But the overwhelming importance of the number-picture, primarily as a means of identification, led, after centuries of its use, to a human invention as clearly a device of man for himself as is the telephone. This was a device for making a primitive individual thinkable as a recognizable and recoverable artificial individual of the kind having numeric quality. This recondite device is measurement. Measurement is an artifice for making a primitive individual conceivable as an artificial individual of the group kind, and so having a number picture. The height of a horse, by use af the unit "a hand," is thinkable as a discrete aggregate and so has a number-picture identifiable by comparison with the standard set of pictures, that is by counting, as say 16.

In Euclid's wonderful Fifth Book a ratio is never a number. Newton, with the purpose of taking in the so-called surds or irrationals of arithmetic and algebra, assumed a ratio to be a number. Any continuity in his number-system comes then from the continuity in the magnitude whose ratio to a chosen unit for

that magnitude is taken. He never gave any arithmetical or algebraic proof of the continuity of any number-system.

Austin, Texas.

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from June-July Number.]

II. Proofs Resulting from Straight-line Properties of the Circle.

XV. Let ABC be $\triangle$ right-angled at C. With either extremity, as B, of the hypotenuse, as a center, and with a radius equal to the hypotenuse, describe a circle. Produce the legs of the $\triangle$ to chords. One of the chords, as DE, will be a diameter.

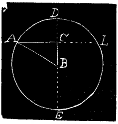


Fig. 11.

Then $AC.CL=DC.CE$, or $b^2=(c-a)(c+a)$.

$\therefore c^2=a^2+b^2$.

XVI. Let ABC be $\triangle$ right-angled at C. With either extremity, as B, of the hypotenuse, as the center, and with a radius equal to the adjacent leg, describe a circle. Produce the hypotenuse to a secant.

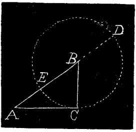


Fig. 12.

Then $AC^2=AE.AD$, or $b^2=(c-a)(c+a)$.

$\therefore c^2=a^2+b^2$.

Note.—This method is given by Richardson in *Runkle's Mathematical Monthly*, No. II, 1859; also by Hoffmann, and, in a slightly different form, by Wipper, the latter stating that the proof is found in "Huberti Rudimenta Algebræ," Wurceb, 1792. It was known to the writers, however, independently of these sources.

XVII. Let ABC be a $\triangle$ right-angled at C.

Case I. When the two legs are unequal.

With C as a center, and with the shorter leg, as BC, as a radius, describe a circle. Produce AC to a secant. Draw CL perpendicular to AB.

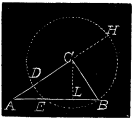


Fig. 13.

Then $AD.AH=AE.AB$,

or $(b-a)(b+a)=c(c-2LB)$.

Substituting for LB any of its equivalents in terms of the sides of the given $\triangle$, and reducing, we get, $c^2=a^2+b^2$.

Case 2. When the two legs are equal.

We easily pass, by the usual method of the theory of limits, from Case 1 to Case 2.

XVIII. Same as in XVII, except that the circle is described with the longer leg, as AC, as a radius. Then, produce all the sides to chords.

Then $AB.BL = BE.BD$,

or $c.BL = (b+a)(b-a)$ .... (1).

Also, $AB : AH :: AC : AL$,

or. $c : 2b :: b : c+BL$,

whence, $c^2 + c.BL = 2b^2$ .... (2).

(1) in (2), $c^2 = a^2 + b^2$.

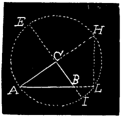


Fig. 14.

When the legs are equal, we pass from the case given as suggested in Case 2 of XVII.

XIX. Same as in XVII, except that both cases are treated alike, and the circle is described with a radius equal to the perpendicular from C to AB. Then produce the legs to secants, and draw CD.

Then $\overline{AD}^2 = AH.AE = b^2 - \overline{CD}^2$;

$\overline{BD}^2 = a^2 - \overline{CD}^2$;

also, $2AD.DB = 2\overline{CD}^2$.

Adding, $c^2 = a^2 + b^2$.

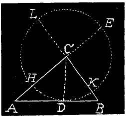


Fig. 15.

XX. Let ABC be a $\triangle$ right-angled at C. Produce either leg, as AC, through C, making $CD = AC$. Join BD. Circumscribe a circle about $\triangle$ ABD, and produce BC to a diameter.

Then $BC^2 = AB.BD - AC.CD$,

or $a^2 = c^2 - b^2$.

$\therefore$ $c^2 = a^2 + b^2$.

XXI. Fig. 16.

$AB.BD = BE.BC$,

or $c^2 = a^2 + a.CE = a^2 + b^2$.

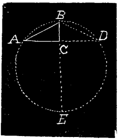


Fig. 16.

NOTE.—The last two are special cases of familiar propositions, and are given by various writers.

[To be Continued.]

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

64. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

If 27 men in 10 days of 7 hours each for \$375 dig a ditch 70 rods long, 25 feet wide, and 4 feet deep, how long a ditch 40 feet wide and 3 feet deep will 15 men dig in 16 days of 9 hours each for \$500 ?

III. Solution by the PROPOSER.

Mr. Gruber's method is all right except the *assumption* that the length of the ditch increases as the price paid. The \$375 pays for 1890 hours' labor ; at the same rate, \$500 would pay for 2520 hours' work. But there are only 2160 hours worked. Hence, the *efficiency* must be increased $\frac{1}{6}$. That is, the ditch will be $66\frac{2}{3}$ rods$\times\frac{7}{6}=77\frac{7}{9}$ rods long.

Or, in another light : Since 1890 hours' labor are worth \$375, 2160 hours' work, at same wages, are worth \$$428\frac{4}{7}$. But they get \$500, an increase of $\frac{1}{6}$ as before.

In this problem the *time* is limited—fixed—hence the only thing that can vary is the *efficiency* of the workmen. And it seems plain that it must increase as the *hourly* price increases—not as the *gross* price. Suppose

2 men in 1 day of 10 hours for \$20 dig x rods, and

3 men in 2 days of 10 hours for \$40 dig y rods. What is the ratio of y to x?

Can the *efficiency*, or productiveness, be found without considering the *hourly* wages ?

66. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Brown adds $m=10\%$ of water to the pure wine he buys, and then sells the mixture at a price $n=10\%$ greater than the cost price of the pure wine. What is his rate per cent. of profit ?

Solution by E. W. MORRELL, Professor of Mathematics in Montpelier Seminary, Montpelier, Vermont.

Let $100\%=$cost of the wine. Then 110% of $110\%=121\%$, the selling price of the mixture. Hence, $121\%-100\%=21\%$, the gain.

67. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

A agreed to work a year for \$300 and a suit of clothes. At the end of five months he left, receiving for his wages \$60 and the clothes. What was the suit worth ?

Solution by P. S. BERG, Larimore, North Dakota.

Since he received $300 and a suit of clothes for a year, for one month he received $25 and $\frac{1}{12}$ suit of clothes, and for five months he received $125 and $\frac{5}{12}$ suit of clothes. He received $60 and the clothes, hence $60+suit of clothes=$125+$\frac{5}{12}$ suit of clothes, or $\frac{7}{12}$ suit=$65. Whence once suit=$$111\frac{3}{7}$.

Also solved by *E. W. MORRELL* and *JAMES F. LAWRENCE.*

68. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

The population of a city is annually increasing $m=2\frac{1}{2}\%$. If the population now is $P=68921$, what was it $n=3$ years ago? At this rate of increase, what will the population be $n=3$ years hence?

Solution by P. S. BERG, Larimore, North Dakota.

Let 100%=what the population was 3 years ago. Then the population at present is $(100\%+2\frac{1}{2}\%)^3$. Hence $(100\%+2\frac{1}{2}\%)^3=68921$. Whence 100%=64000, the population 3 years ago. In 3 years hence the population will be $(100\%+2\frac{1}{2}\%)^3$ of 68921, or 74220.378765625.

69. Proposed by EDGAR M. JOHNSON, Professor of Mathematics, Emory College, Oxford, Georgia.

Every man in a certain group belongs to at least one of these classes: Methodists, Democrats, Farmers. In the group there are 10 Methodists, 12 Democrats. 13 Farmers; 3 men who are Methodists and Democrats, 4 who are Democrats and Farmers, 5 who are Methodists and Farmers. Finally, there are 2 men who are at the same time Methodists, Democrets and Farmers. Required the number of men in the group.

I. Solution by J. C. CORBIN, Pine Bluff, Arkansas.

Using obvious abbreviations, we can form the following table in which each small letter denotes a man:

Methodists.	Democrats.	Farmers.
a, *b*	*a*, *b*	*a*, *b*
c, *d*, *e*, *f*, *g*	*h*, *i*, *j*, *k*	*h*, *i*, *j*, *k*
l, *m*, *n*	*l*, *m*, *n*	*r*, *s*
	o, *p*, *q*	

Counting each letter once only, gives 19; 10 in the first column, 12 in the second column, and 13 in the third column.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas, and FREDERICK R. HONEY, Ph. B., New Haven, Connecticut.

Methodists.	Democrats.	Farmers.	Total.
3	3	0	3
0	4	4	4
5	0	5	5
2	2	2	2
10	9	11	14
0	3	2	5
10	12	13	19

∴ 19 men in the group.

Also solved by *E. W. MORRELL*, *JAMES F. LAWRENCE* and *P. S. BERG.*

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

66. Proposed by A. H. BELL, Box 184, Hillsboro, Illinois.

Solve the equations:

$$a^2x=(2x^2-a^2)\sqrt{x^2+y^2}\ \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$b^2y=(2y^2-b^2)\sqrt{x^2+y^2}\ \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

I. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let $x=r\cos\theta$, $y=r\sin\theta$. Then the equations become

$$a^2\cos\theta=2r^2\cos^2\theta-a^2\ \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$b^2\sin\theta=2r^2\sin^2\theta-b^2\ \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Eliminating r^2 from (1) and (2) we get

$$\frac{b^2(1+\sin\theta)}{1+\cos\theta}=a^2\tan^2\theta\ \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Now $\sin\theta=\dfrac{2\tan\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}$, $\cos\theta=\dfrac{1-\tan^2\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}$. $\therefore\ b^2(1+\tan\frac{1}{2}\theta)^2=2a^2\tan^2\theta$,

$$\text{or } b(1+\tan\tfrac{1}{2}\theta)=\pm\sqrt{2}\,a\tan\theta\ \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Let $z=\tan\frac{1}{2}\theta$; then (4) becomes

$$z^3+z^2-\left(1\mp\frac{2\sqrt{2a}}{b}\right)z=1\ \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Let $u=z-\frac{1}{3}$; then (5) becomes

$$u^3-\tfrac{2}{3}\left(2\mp\frac{3\sqrt{2a}}{b}\right)u=\tfrac{2}{27}\left(8\pm\frac{9\sqrt{2a}}{b}\right)\ \ldots\ldots\ldots\ldots\ldots(6).$$

When a and b are known we can find u from (6), after which z and r and finally x and y become known.

II. Solution by HENRY HEATON, M. Sc., Atlantic, Iowa.

As in preceding solution,

$$a^2\cos\theta=2r^2\cos^2\theta-a^2\ \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$b^2\sin\theta=2r^2\sin^2\theta-b^2\ \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

$$\therefore\ 2r^2\cos^2\theta=a^2(1+\cos\theta)\ldots\ldots(5),\ \text{and}\ 2r^2\sin^2\theta=b^2(1+\sin\theta)\ldots\ldots(6).$$

Dividing (5) by (4), $\tan^2\theta=\dfrac{b^2}{a^2}\left(\dfrac{1+\sin\theta}{1+\cos\theta}\right)=\dfrac{b^2}{a^2}\left(\dfrac{\sec\theta+\tan\theta}{\sec\theta+1}\right)\ldots\ldots(7).$

$$\therefore\ a^2\tan^2\theta\sec\theta+a^2\tan^2\theta=b^2\sec\theta+b^2\tan\theta\ldots\ldots(8).$$

$$\therefore\ (a^2\tan^2\theta-b^2)\sec\theta=b^2\tan\theta-a^2\tan^2\theta\ldots\ldots(9).$$

Squaring (9) and substituting for $\sec^2\theta$ its value $1+\tan^2\theta$, performing operations indicated and arranging with reference to $\tan\theta$,

$$a^4\tan^6\theta-2a^2b^2\tan^4\theta+2a^2b^2\tan^3\theta-2a^2b^2t^2+b^4=0\ldots\ldots(10).$$

Transposing the three middle terms and subtracting $2a^2b^2\tan^3\theta$,

$$a^4\tan^6\theta-2a^2b^2\tan^3\theta+b^4=2a^2b^2(\tan^4\theta-2\tan^3\theta+\tan^2\theta)\ \ldots\ldots(11).$$

Extracting square root, $a^2\tan^3\theta-b^2=\pm ab(\tan^2\theta-\tan\theta)\sqrt{2}\ldots\ldots(12).$

Whence $\tan\theta=\dfrac{\pm d\sqrt{2}}{3}+\dfrac{1}{3a}\left[\dfrac{b^2}{2}(9a\pm4b\sqrt{2})\right.$

$$\left.+\frac{3b}{2}(\pm24a^3b\sqrt{2}-39a^2b^2\pm24ab^3\sqrt{2})^{\frac{1}{2}}\right]^{\frac{1}{3}}$$

$$+\frac{1}{3a}\left[\frac{b^2}{2}(9a\pm4b\sqrt{2})-\frac{3b}{2}(\pm24a^3b\sqrt{2}-39a^2b^2\pm24ab^3\sqrt{2})^{\frac{1}{2}}\right]^{\frac{1}{3}}.$$

From equation (5), $x=a\sqrt{\dfrac{1+\cos\theta}{2}}=a\cos\frac{1}{2}\theta$, and from equation (6)

$$y=b\sqrt{\frac{1+\sin\theta}{2}}=b\cos(\tfrac{1}{4}\pi-\tfrac{1}{2}\theta)=\frac{b}{\sqrt{2}}(\cos\tfrac{1}{2}\theta-\sin\tfrac{1}{2}\theta).$$

If $a=b$, from (12), $\tan\theta=1$ or $\dfrac{\pm\sqrt{2}-1}{2}\pm\frac{1}{2}(\pm2\sqrt{2}-1)^{\frac{1}{2}}$.

III. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

Dividing (1) by (2) and putting $y=tx$, we obtain

$$x^2=\frac{a^2b^2(1-t)}{2t(a^2t-b^2)},\ \text{and then}\ y^2=\frac{a^2b^2t(1-t)}{2(a^2t-b^2)}.$$

Substituting these in (1), we obtain finally the equation

$$t^6-\frac{2b^2}{a^2}t^4+\frac{2b^2}{a^2}t^3-\frac{2b^2}{a^2}t^2+\frac{b^4}{a^4}=0.$$

Solving this for numerical values of a and b, we get the values of x and y from the above expressions.

The same equation may be arrived at by putting $x=r\cos\theta$, $y=r\sin\theta$. The given equation then changes into $a^2\cos\theta=2r^2\cos^2\theta-a^2$; $b^2\sin\theta=2r^2\sin^2\theta-b^2$. Adding, we get $a^2\cos\theta+b^2\sin\theta=2r^2-(a^2+b^2)$, whence $r^2=\frac{1}{2}[a^2\cos\theta+b^2\sin\theta+a^2+b^2]$. Also, $r^2=\frac{1}{2}.\dfrac{a^2\cos\theta+a^2}{\cos^2\theta}$.

Equalizing, changing into the tangent function, the latter being denoted by t, we obtain the same equation as above.

IV. Solution by H. C. WILKES, Skull Run, West Virginia.

Putting $x^2+y^2=s^2$; then from (1), $a^2(x+s)=2sx^2$, and from (2), $b^2(y+s)=2sy^2$. Any rational value for s will give integral [?] fractional values for a^2 and b^2. Let $s=5$, $a^2=45/4$, and $b^2=160/9$; $s=13$, $a^2=325/9$, and $b^2=3744/25$; $s=17$, $a^2=2176/25$, and $b^2=3825/16$.

67. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.

Prove that $\cos\frac{n\pi}{7}+\cos\frac{3n\pi}{7}+\cos\frac{5n\pi}{7}=\frac{1}{2}$ or $-\frac{1}{2}$, according as n is *odd* or *even*, [and not a multiple of 7].

I. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

$$\sin\frac{2n\pi}{7}=2\sin\frac{n\pi}{7}\cos\frac{n\pi}{7}.$$

$$\sin\frac{4n\pi}{7}-\sin\frac{2n\pi}{7}=2\sin\frac{n\pi}{7}\cos\frac{3n\pi}{7}.$$

$$\sin\frac{6n\pi}{7}-\sin\frac{4n\pi}{7}=2\sin\frac{n\pi}{7}\cos\frac{5n\pi}{7}.$$

$$\therefore\cos\frac{n\pi}{7}+\cos\frac{3n\pi}{7}+\cos\frac{5n\pi}{7}=\frac{\sin\frac{6}{7}n\pi}{2\sin\frac{1}{7}n\pi}=\frac{\sin(n\pi-\frac{1}{7}n\pi)}{2\sin\frac{1}{7}n\pi}$$

$$=-\tfrac{1}{2}\cos n\pi=\pm\tfrac{1}{2},\text{ according as } n \text{ is odd or even.}$$

II. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

Employing the well-known formula

$$\sum_{n=1}^{n=n}\cos[a+(n-1)b]=\frac{\cos[a+\frac{1}{2}(n-1)b]\sin\frac{1}{2}nb}{\sin\frac{1}{2}b},$$

and putting $b=2a$, $n=3$, we have $\cos a+\cos 3a+\cos 5a=\dfrac{\cos 3a\sin 3a}{\sin a}=\frac{1}{2}\dfrac{\sin 6a}{\sin a}$.

But either $\frac{1}{2}\dfrac{\sin 6a}{\sin a}=\frac{1}{2}\dfrac{\sin 6a+\sin a}{\sin a}-\frac{1}{2}=\dfrac{\sin\frac{7}{2}a.\cos\frac{5}{2}a}{\sin a}-\frac{1}{2}$,

or, $\frac{1}{2}\dfrac{\sin 6a}{\sin a}=\frac{1}{2}\dfrac{\sin 6a-\sin a}{\sin a}+\frac{1}{2}=\dfrac{\cos\frac{7}{2}a.\sin\frac{5}{2}a}{\sin a}+\frac{1}{2}$.

Putting $a=\frac{1}{7}n\pi$, we get in the former case $\dfrac{\sin\frac{1}{2}n\pi\cos\frac{5}{14}n\pi}{\sin\frac{1}{7}n\pi}-\frac{1}{2}$, and in the latter $\dfrac{\cos\frac{1}{2}n\pi\sin\frac{5}{14}n\pi}{\sin\frac{1}{7}n\pi}+\frac{1}{2}$. If n is *even*, $\sin\frac{1}{2}n\pi=0$, if *odd*, $\cos\frac{1}{2}n\pi=0$.

Q. E. D.

III. Solution by OTTO O. CLAYTON, A. B., Fowler, Indiana.

Unite 1st and 3rd terms of the left member ; then by factoring, we have,

$$(2\cos\tfrac{2}{7}n\pi+1)\cos\tfrac{3}{7}n\pi=\tfrac{1}{2}\text{ or }-\tfrac{1}{2}.$$

Substituting for $(2\cos\frac{2}{7}n\pi+1)$, we have $\dfrac{\sin\frac{3}{7}n\pi\cos\frac{3}{7}n\pi}{\sin\frac{1}{7}n\pi}=\frac{1}{2}$ or $-\frac{1}{2}$, from which $\frac{1}{2}\dfrac{\sin\frac{6}{7}n\pi}{\sin\frac{1}{7}n\pi}=\frac{1}{2}.\dfrac{\sin-\frac{1}{7}n\pi}{\sin\frac{1}{7}n\pi}=\frac{1}{2}$ or $-\frac{1}{2}$.

This being an identical equation the problem is proved ; for ratio

$$\frac{\sin-\frac{1}{7}n\pi}{\sin\frac{1}{7}n\pi}=1\text{ or }-1,\text{ according as } n \text{ is odd or even.}$$

IV. Solution by JOHN B. FAUGHT, A. M., Instructor in Mathematics in Indiana University, Bloomington. Indiana.

The equation (1), $(\cos\theta+i\sin\theta)^7=-1$, *i. e.*, $\cos 7\theta+i\sin 7\theta=-1$, is clearly satisfied when θ has either of the following values : $\frac{1}{7}\pi$, $\frac{3}{7}\pi$, $\frac{5}{7}\pi$, $\frac{7}{7}\pi$, $\frac{9}{7}\pi$, $\frac{11}{7}\pi$ and $\frac{13}{7}\pi$.

$\therefore$ (2), $(\cos n\theta+i\sin n\theta)^7=(-1)^n$ is satisfied by $\theta=\frac{1}{7}\pi$, $\frac{3}{7}\pi$, $\frac{5}{7}\pi$, $\frac{7}{7}\pi$, $\frac{9}{7}\pi$, $\frac{11}{7}\pi$ or $\frac{13}{7}\pi$, or $n\theta=\frac{1}{7}n\pi$, $\frac{3}{7}n\pi$, $\frac{5}{7}n\pi$, $\frac{7}{7}n\pi$, $\frac{9}{7}n\pi$, $\frac{11}{7}n\pi$ or $\frac{13}{7}n\pi$.

But (3), $(\cos n\theta+i\sin n\theta)^7=\cos^7 n\theta+7i\cos^6 n\theta\sin n\theta-21\cos^5 n\theta\sin^2 n\theta$ $-35i\cos^4 n\theta\sin^3 n\theta+35\cos^3 n\theta\sin^4 n\theta+21i\cos^2 n\theta\sin^5 n\theta-7\cos n\theta\sin^6 n\theta-i\sin^7 n\theta$ $=(-1)^n$.

$\therefore$ (4), $\cos^7 n\theta-21\cos^5 n\theta\sin^2 n\theta+35\cos^3 n\theta\sin^4 n\theta-7\cos n\theta\sin^6 n\theta=(-1)^n$.

Or (5), $64\cos^7 n\theta-112\cos^5 n\theta+56\cos^3 n\theta-7\cos n\theta-(-1)^n=0$, of which

$\cos\frac{1}{7}n\pi$, $\cos\frac{3}{7}n\pi$, $\cos\frac{5}{7}n\pi$, $\cos\frac{7}{7}n\pi$, $\cos\frac{9}{7}n\pi$, $\cos\frac{11}{7}n\pi$ and $\cos\frac{13}{7}n\pi$ are the ratio.

Now $\cos\frac{7}{7}n\pi=\mp1$, according as n is *odd* or *even*, and $\cos\frac{13}{7}n\pi=\cos\frac{1}{7}n\pi$; $\cos\frac{9}{7}n\pi=\cos\frac{5}{7}n\pi$.

Hence we have (6), $(\cos n\theta\pm1)(64\cos^6 n\theta\mp64\cos^5 n\theta-48\cos^4 n\theta\pm48\cos^3 n\theta+8\cos^2 n\theta\mp\cos n\theta+1)=0$, according as n is *odd* or *even*.

$\therefore$ (7), $2(\cos\frac{1}{7}n\pi+\cos\frac{3}{7}n\pi+\cos\frac{5}{7}n\pi)=\pm\frac{64}{64}=\pm1$. Or $\cos\frac{1}{7}n\pi+\cos\frac{3}{7}n\pi+\cos\frac{5}{7}n\pi=\pm\frac{1}{2}$, according as n is *odd* or *even*.

We might deduce a number of equally interesting results, thus,

$$(\cos\tfrac{1}{7}n\pi.\cos\tfrac{3}{7}n\pi.\cos\tfrac{5}{7}n\pi)^2=\tfrac{1}{64}.$$

$\therefore$ $\cos\frac{1}{7}n\pi.\cos\frac{3}{7}n\pi.\cos\frac{5}{7}n\pi=\pm\frac{1}{8}$, when n is either *odd* or *even*, etc.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

63. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, P. O., University of Mississippi.

A rectangular hyperbola cannot be cut from a right circular cone if the angle at its vertex is less than a right angle.

II. Solution by F. M. McGAW, A. M., Professor of Mathematics, Bordentown Military Institute, Bordentown, New Jersey.

Assume axes of coördinates at right angles.

(a) The equation of the surface of a cone with axis of z as axis of cone, and origin at the vertex of cone is

$$x^2+y^2-z^2\tan^2\tfrac{1}{2}v=0\ldots\ldots\ldots\ldots(1)$$

where v=angle at vertex.

(b) The equation of a plane to same axes and origin as above, in terms of its direction cosines and perpendicular from origin is

$$lx+my+nz=\phi\ldots\ldots\ldots\ldots(2).$$

Eliminate z between (1) and (2), and then we have the conic

$$(n^2-l^2\alpha^2)x^2-2lmxy\alpha^2+y^2(n^2-m^2\alpha^2)+2pl\alpha^2x+2pm\alpha^2y-p^2\alpha^2=0\,..(3)$$

in which α is substituted for $\tan\frac{1}{2}v$.

In order that this conic may be an equilateral hyperbola, the angle between its asymptotes

$$(n^2-l^2\alpha^2)x^2-2lm\alpha^2xy+(n^2-m^2\alpha^2)y^2=0$$

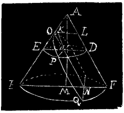
must be a right angle, the condition for which is (for rectangular axes)

$$n^2-l^2\alpha^2+n^2-m^2\alpha^2=0,\text{ or }\alpha^2=2n^2/l^2+m^2\ \ldots\ldots\ldots\ldots(4).$$

Now, in order that the plane above considered shall cut out the hyperbola, the angle whose direction cosine is n must be less than $\frac{1}{2}v$; that is to say, l and m must both be less than n. Hence, l^2+m^2 is necessarity less than n^2+n^2 or $2n^2$; or the fraction (4) is an improper fraction, whence $\alpha^2(=\tan^2\frac{1}{2}v)$ is greater than unity. This establishes that $\frac{1}{2}v$ is greater than 45°, and v is greater than 90°. Q. E. D.

III. Solution by GEORGE LILLEY, LL. D., Portland, Oregon.

Let ABF be a section of the cone made by the plane of the paper passing through its axis AM; $OPQNH$ any section of the cone made by a plane perpendicular to the plane ABF. Pass a plane through P at right angles to AM cutting the plane ABF in DE. Draw OL parallel to BF, and HK parallel to AF. Let $\angle MAB=\alpha$, $\angle AOH=\theta$, $AO=c$, $OH=x$, and $HP=y$.



$$HP^2=HD\times HE.\quad HE=\frac{x\sin\theta}{\cos\alpha},$$

$$DH=LK=2c\sin\alpha-\frac{x\sin(\theta+2\alpha)}{\cos\alpha}.$$

$$\therefore\ y^2=\frac{2c\sin\theta\sin\alpha}{\cos\alpha}x-\frac{\sin\theta\sin(\theta+2\alpha)}{\cos^2\alpha}x^2.$$

The section represented by the equation is any hyperbola when $\theta+2\alpha$ is greater than 180°. Comparing the equation with $y^2=\dfrac{2b^2}{a}x+\dfrac{b^2}{a^2}x^2$, we have

$$\frac{2b^2}{a}=\frac{2c\sin\alpha\sin\theta}{\cos\alpha},\quad\frac{b^2}{a^2}=\frac{\sin\theta\sin(\theta+2\alpha)}{\cos^2\alpha}.$$

$$\therefore\ 2a=\frac{c\sin2\alpha}{\sin(\theta+2\alpha)},\quad b^2=\frac{c^2\sin^2\alpha}{\sin(\theta+2\alpha)}.$$

$e^2=\frac{a^2+b^2}{a^2}=\frac{1-\sin^2(\theta+\alpha)-2\sin^2\alpha}{\cos^2\alpha}$, where e is the excentricity of the hyperbola. Or $1=\frac{1-\sin^2(\theta+\alpha)-2\sin^2\alpha}{e^2\cos^2\alpha}$.

$e^2\cos^2\alpha$ must not be greater than unity. But $e^2=2$; therefore, $\cos^2\alpha$ must not be greater than $\frac{1}{2}$, and α must not be less than 45°. Hence, the angle at the vertex of the cone must not be less than a right angle; therefore, it is greater than a right angle.

It may, however, be equal to that angle.

Note on the angle between the asymptotes of the hyperbola.

Let $\phi=$the angle between the asymptotes, and we have $\sec\frac{1}{2}\phi=e$, where e is the excentricity of the hyperbola.

$\sec^2\frac{1}{2}\phi\cos^2\alpha=e^2\cos^2\alpha$, or $\frac{\cos^2\alpha}{\cos^2\frac{1}{2}\phi}=e^2\cos^2\alpha$, but $e^2\cos^2\alpha$ must not be be greater than unity, see solution of problem 63. Hence, $\cos\frac{1}{2}\phi$ must not be less than $\cos\alpha$ and α must not be less than $\frac{1}{2}\phi$; or the angle at the vertex of the right circular cone, from which the hyperbola is cut, must not be less than the angle between the asymptotes. *It may, however, be equal to that angle.*

IV. Solution by W. H. CARTER, Professor of Mathematics in Centenary College of Louisiana, Jackson, Louisiana.

If the axis of the cone be the axis z; h, the distance of the vertex from the origin, and θ the semi-angle at the vertex, the equation of the cone is

$$(x^2+y^2)\tan^2(90^\circ-\theta)=(h-z)^2.$$

The section of this cone made by a plane through the axis y is a conic section, and if the angle which the plane makes with xy be ϕ and the curve of intersection be referred to axes in its own plane, its equation is

$$y^2\tan^2(90^\circ-\theta)+x^2\cos^2\phi[\tan^2(90^\circ-\theta)-\tan^2\phi]+2hx\sin\phi-h^2=0.$$

If this is a rectangular hyperbola, then

$$\tan^2(90^\circ-\theta)=\cos^2\phi[\tan^2\phi-\tan^2(90^\circ-\theta)].$$

$$\therefore\ \tan\phi=\pm\frac{\sqrt{2\sin(90^\circ-\theta)}}{\sqrt{\cos^2(90^\circ-\theta)}}.\quad \text{But } \phi \text{ is real.}$$

$$\therefore\ 90^\circ-\theta<45\ .\quad \therefore\ -\theta<-45\ .\quad \therefore\ 2\theta>90\ .$$

An hyperbola may also be cut from this cone by a plane parallel to the axis z, Its equation then is, if the cutting plane is $y=a$,

$$(x^2+a^2)\tan^2(90^\circ-\theta)=(h-z)^2.$$

If this be a rectangular hyperbola,

$$\tan^2(90^\circ-\theta)=1 \qquad \tan(90^\circ-\theta)=1 \qquad (-1 \text{ makes } \theta \text{ negative}).$$

$$\therefore\ 90^\circ-\theta=45^\circ. \quad \therefore\ \theta=45^\circ. \quad \therefore\ 2\theta=90^\circ.$$

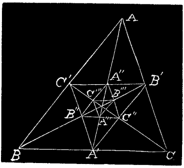
This problem was also solved in an excellent manner by *G. B. M. ZERR.*

64. Proposed by WILLIAM E. HEAL, Member of the London Mathematical Society, and Treasurer of Grant County, Marion, Indiana.

Let the bisectors of the angles A, B, C of a triangle meet the sides opposite A, B, C in A', B', C'. Let AA', BB', CC' meet the sides of the triangle $A'B'C'$ in A'', B'', C''. Let this process continue indefinitely. Express the sides and angles of the triangle $A^{(m)}B^{(m)}C^{(m)}$ in terms of the sides and angles of the original triangle ABC.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Using trilinear coordinates we have $\beta-\gamma=0$, $\gamma-\alpha=0$, $\alpha-\beta=0$ for the equations to AA', BB', CC' respectively.



$$\left(0,\ \frac{2\triangle}{b+c},\ \frac{2\triangle}{b+c}\right),\ \left(\frac{2\triangle}{a+c},\ 0,\ \frac{2\triangle}{a+c}\right),$$

$$\left(\frac{2\triangle}{a+b},\ \frac{2\triangle}{a+b},\ 0\right)$$

are the coordinates of A', B', C' respectively.

$\therefore\ \alpha+\beta-\gamma=0$, $\alpha+\gamma-\beta=0$, $\beta+\gamma-\alpha=0$ are the equations to $A'B'$, $A'C'$, $B'C'$ respectively.

$$\left(\frac{4\triangle}{2a+b+c},\ \frac{2\triangle}{2a+b+c},\ \frac{2\triangle}{2a+b+c}\right),\ \left(\frac{2\triangle}{a+2b+c},\ \frac{4\triangle}{a+2b+c},\ \frac{2\triangle}{a+2b+c}\right),$$

$$\left(\frac{2\triangle}{a+b+2c},\ \frac{2\triangle}{a+b+2c},\ \frac{4\triangle}{a+b+2c}\right)$$

are the coordinates of A'', B'', C'' respectively.

$\therefore\ \alpha+\beta-3\gamma=0$, $\alpha+\gamma-3\beta=0$, $\beta+\gamma-3\alpha=0$ are the equations to $A''C''$, $A''C''$, $B''C''$ respectively.

$$\left(\frac{4\triangle}{2a+3b+3c},\ \frac{6\triangle}{2a+3b+3c},\ \frac{6\triangle}{2a+3b+3c}\right),\ \left(\frac{6\triangle}{3a+2b+3c},\ \frac{4\triangle}{3a+2b+3c},\right.$$

$$\frac{6\triangle}{3a+2b+3c}\Big),\ \Big(\frac{6\triangle}{3a+3b+2c},\ \frac{6\triangle}{3a+3b+2c},\ \frac{4\triangle}{3a+3b+2c}\Big)$$

are the coordinates of A''', B''', C''' respectively.

$\therefore$ $3\alpha+3\beta-5\gamma=0$, $3\alpha+3\gamma-5\beta=0$, $3\beta+3\gamma-5\alpha=0$ are the equations to $A'''B'''$, $A'''C'''$, $B'''C'''$ respectively.

$$\Big(\frac{12\triangle}{6a+5b+5c},\ \frac{10\triangle}{6a+5b+5c},\ \frac{10\triangle}{6a+5b+5c}\Big),\ \Big(\frac{10\triangle}{5a+6b+5c},\ \frac{12\triangle}{5a+6b+5c},$$

$$\frac{10\triangle}{5a+6b+5c}\Big),\ \Big(\frac{10\triangle}{5a+5b+6c},\ \frac{10\triangle}{5a+5b+6c},\ \frac{12\triangle}{5a+5b+6c}\Big)$$

are the coordinates of A'''', B'''', C'''' respectively.

$\therefore$ $5\alpha+5\beta-11\gamma=0$, $5\alpha+5\gamma-11\beta=0$, $5\beta+5\gamma-11\alpha=0$ are the equations to $A''''B''''$, $A''''C''''$, $B''''C''''$ respectively.

In what follows, the upper signs are used for m odd, and the lower for m even. The mth term of the series 1, 3, 5, 11, 21, 43, 85, etc., is $\frac{1}{3}(2^m\pm1)$.

$$\therefore\ \Big(\frac{4\triangle(2^{m-1}\mp1)}{2(2^{m-1}\mp1)a+(2^m\pm1)(b+c)},\ \frac{2\triangle(2^m\pm1)}{2(2^{m-1}\mp1)a+(2^m\pm1)(b+c)},$$

$$\frac{2\triangle(2^m\pm1)}{2(2^{m-1}\mp1)a+(2^m\pm1)(b+c)}\Big),\ \Big(\frac{2\triangle(2^m\pm1)}{2(2^{m-1}\mp1)b+(2^m\pm1)(a+c)},$$

$$\frac{4\triangle(2^{m-1}\mp1)}{2(2^{m-1}\mp1)b+(2^m\pm1)(a+c)},\ \frac{2\triangle(2^m\pm1)}{2(2^{m-1}\mp1)b+(2^m\pm1)(a+c)}\Big),$$

$$\Big(\frac{2\triangle(2^m\pm1)}{2(2^{m-1}\mp1)c+(2^m\pm1)(a+b)},\ \frac{2\triangle(2^m\pm1)}{2(2^{m-1}\mp1)c+(2^m\pm1)(a+b)},$$

$$\frac{4\triangle(2^{m-1}\mp1)}{2(2^{m-1}\mp1)c+(2^m\pm1)(a+b)}\Big)$$

are the coordinates of A^m, B^m, C^m respectively.

$\therefore$ $\frac{1}{3}(2^m\pm1)(\alpha+\beta)-\frac{1}{3}(2^{m+1}\mp1)\gamma=0$, $\frac{1}{3}(2^m\pm1)(\alpha+\gamma)-\frac{1}{3}(2^{m+1}\mp1)\beta=0$, $\frac{1}{3}(2^m\pm1)(\beta+\gamma)-\frac{1}{3}(2^{m+1}\mp1)\alpha=0$, (1, 2, 3) are the equations to A^mB^m, A^mC^m, B^mC^m respectively.

From (1) and (2), (1) and (3), (2) and (3) respectively, we get

$$\tan A^m=\frac{3\{2^{m+1}(2^{m-1}\mp1)\sin A+2^m(2^m\pm1)(\sin B+\sin C)\}}{3(2^{2m}-1)+2(5.2^{2m-1}\mp2^m+1)\cos A-2(2^m\pm1)(2^{m-1}\mp1)(\cos B+\cos C)}$$

$$\tan B^{m}=\frac{3\{2^{m+1}(2^{m-1}\mp1)\sin B+2^{m}(2^{m}\pm1)(\sin A+\sin C)\}}{3(2^{2m}-1)+2(5.2^{2m-1}\mp2^{m}+1)\cos B-2(2^{m}\pm1)(2^{m-1}\mp1)(\cos A+\cos C)}$$

$$\tan C^{m}=\frac{3\{2^{m+1}(2^{m-1}\mp1)\sin C+2^{m}(2^{m}\pm1)(\sin A+\sin B)\}}{3(2^{2m}-1)+2(5.2^{2m-1}\mp2^{m}+1)\cos C-2(2^{m}\pm1)(2^{m-1}\mp1)(\cos A+\cos B)}.$$

Let $A=$ area of $A^{(m)}B^{(m)}C^{(m)}$, $p=$ perpendicular from $C^{(m)}$ on $A^{(m)}B^{(m)}$.

$$\therefore\ A=[27abc\Delta.2^{m}]\div[\{2(2^{m-1}\mp1)a+(2^{m}\pm1)(b+c)\}$$
$$\{2(2^{m-1}\mp1)b+(2^{m}\pm1)(a+c)\}\{2(2^{m-1}\mp1)c+(2^{m}\pm1)(a+b)\}]$$

$$p=[9\Delta.2^{m+1}]\div[\{(2^{m}\pm1)(a+b)+2(2^{m-1}\mp1)c\}$$
$$\sqrt{3(2^{2m+1}+1)+2(2^{m}\pm1)(2^{m+1}\mp1)(\cos A+\cos B)-2(2^{m}\pm1)^{2}\cos C}].$$

But $A=\frac{1}{2}pA^{(m)}B^{(m)}$. $\therefore\ A^{(m)}B^{(m)}=2A/p$.

$\therefore\ A^{(m)}B^{(m)}$

$$=\frac{3abc\sqrt{3(2^{2m+1}+1)+2(2^{m}\pm1)(2^{m+1}\mp1)(\cos A+\cos B)-2(2^{m}\pm1)^{2}\cos C}}{\{2(2^{m-1}\mp1)a+(2^{m}\pm1)(b+c)\}\{2(2^{m-1}\mp1)b+(2^{m}\pm1)(a+c)\}}$$

$$A^{(m)}C^{(m)}=\frac{3abc\sqrt{3(2^{2m+1}+1)+2(2^{m}\pm1)(2^{m+1}\mp1)(\cos A+\cos C)-2(2^{m}\pm1)^{2}\cos B}}{\{2(2^{m-1}\mp1)a+(2^{m}\pm1)(b+c)\}\{2(2^{m-1}\mp1)c+(2^{m}\pm1)(a+b)\}}$$

$$B^{(m)}C^{(m)}=\frac{3abc\sqrt{3(2^{2m+1}+1)+2(2^{m}\pm1)(2^{m+1}\mp1)(\cos B+\cos B)-2(2^{m}\pm1)^{2}\cos A}}{\{2(2^{m-1}\mp1)b+(2^{m}\pm1)(a+c)\}\{2(2^{m-1}\mp1)c+(2^{m}\pm1)(a+b)\}}$$

All principles necessary to understand the above solution will be found in the chapter on "Trilinear Coordinates" in Todhunter's Conic Sections.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

53. Proposed by O. D. SMITH, A. M., Professor of Mathematics, Alabama Polytechnic Institute, Auburn, Alabama.

Solve the differential equation, $dy/dx=y(x-y)/x(x+y)$, and show that $x=y\log(xy)$.

I. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland; C. W. M. BLACK, Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts; and P. S. BERG, Larimore, North Dakota.

Clearing of fractions we obtain after transposing two terms

$$y(xdy+ydx)=x(ydx-xdy).$$

Dividing by y^2, we have

$$xdy+ydx=xy.\frac{ydx-xdy}{y^2}, \text{ or, } d(xy)=xy.d\left(\frac{x}{y}\right). \quad \therefore \frac{d\,(xy)}{xy}=d\left(\frac{x}{y}\right).$$

Integrating, $\log(axy)=(x)\div(y)$, whence $x=y\log(axy)$.

The result given is not general enough, the constant having been left out of consideration.

II. Solution by W. W. LANDIS, A. M., Department of Mathematics and Astronomy in Dickinson College, Carlisle, Pennsylvania; F. M. McGAW, A. M., Professor of Mathematics, Bordentown Military Institute, Bordentown, New Jersey; J. OWEN MAHONEY, B. E., Graduate Fellow and Assistant in Mathematics, Vanderbilt University, Nashville, Tennessee; G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas; and A. H. HOLMES, Brunswick, Maine.

Let $y=vx$, then the equation becomes

$$v+\frac{xdv}{dx}=\frac{x(1-v)}{1+v}, \text{ or } 2v^2+\frac{x(1+v)dv}{dx}=0. \quad \therefore \frac{1+v}{2v^2}dv+\frac{dx}{x}=0.$$

The variables are separable, whence

$$\frac{dv}{2v^2}+\frac{dv}{2v}+\frac{dx}{x}=0. \quad \text{Integrating, } \frac{1}{2v}=\tfrac{1}{2}\log v+\log x.$$

$\therefore 1/v=\log v+\log x^2=\log(vx^2)$.

$\therefore x/y=\log(xy)$, or $x=y\log(xy)$, when no constant is added, or $x=y\log(xy)+Cy$ where C is an arbitrary constant.

III. Solution by M. C. STEVENS, M. A., Mathematical Department, Purdue University, Lafayette, Indiana; HENRY HEATON, M. Sc., Atlantic, Iowa; JOHN B. FAUGHT, A. M., Instructor in Mathematics in Indiana University, Bloomington, Indiana; and J. C. GREGG, A. M., Brazil, Indiana.

Put $y=vx$ and we have

$$v+x\frac{dv}{dx}=\frac{vx^2-v^2x^2}{x^2+vx^2}=\frac{v-v^2}{1+v}=v-\frac{2v^2}{1+v}. \quad \text{Whence, } \frac{1+v}{v^2}dv+\frac{2dx}{x}=0.$$

Integrating, $-1/v+\log(vx^2)+C=0$, or $x/y=\log(xy)+C$, and $x=y\log(xy)+Cy$.

The C should not be omitted unless the conditions of the question giving rise to the equation are such as to make it zero.

IV. Solution by H. C. WHITTAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

Let $y=x^pv^q$ and substitute in the given equation and we obtain

$$\frac{dv}{dx}=\frac{(1-p)v-(1+p)x^{p-1}v^{q+1}}{q^r(1+x^{p-1}v^q)}.$$

This will reduce to a simple form if we take $p=1$ and $q=-1$, giving

$$\frac{dv}{dx}=\frac{2}{x(1+v^{-1})}, \text{ or } dv(1+v^{-1})=2x^{-1}dx.$$

$v+\log v=\log x^2$; $x/y+\log(x/y)=\log x^2$.

$x/y=\log x^2-\log(x/y)=\log(xy)$, whence $x=y\log(xy)$.

54. Proposed by J. SCHEFFER, A. M., Hagerstown, Maryland.

A certain solid has a square, side$=a$, for its base, and all parallel sections are squares, the two sections through the middle points of the opposite side of the square are semi-circles, however. Find surface, volume, and center of gravity of each.

I. Solution by HENRY HEATON, M. Sc., Atlantic, Iowa.

The length of a side of a parallel section distant x from the base is $(a^2-4x^2)^{\frac{1}{2}}$. If dx be the distance between two parallel sections, the distance between two corresponding sides is $adx/(a^2-4x^2)^{\frac{1}{2}}$. Hence the surface

$$S=4\int_0^{\frac{1}{2}a} adx=2a^2 \text{ ; the volume } V=\int_0^{\frac{1}{2}a}(a^2-4x^2)dx=\tfrac{1}{3}a^3 \text{ ;}$$

the distance of the center of gravity of the surface from the base is

$$\frac{1}{2a^2}\int_0^{\frac{1}{2}a} axdx=\tfrac{1}{8}a \text{ ;}$$

and the distance of the center of gravity of the volume from the base is

$$\frac{3}{a^3}\int_0^{\frac{1}{2}a} x(a^2-4x^2)dx=\tfrac{3}{16}a.$$

II. Solution by J. C. NAGLE, M. A., M. C. E., Professor of Civil Engineering in the State A. M. College, College Station, Texas.

Take the intersection of the planes of the circular sections as the axis of z, the origin being in the center of the base. Then since the radius of each circle is $\frac{1}{2}a$ we shall have for the projection of one fourth of the elementary area intercepted between two planes parallel to the base, and distant dz from each other, upon the plane of one of the circles,

$$dS.\cos\theta=\sqrt{\tfrac{1}{4}a^2-z^2}.dz,$$

where θ is the angle made by this elementary area with the plane of projection.

But $\cos\theta=\dfrac{\sqrt{\frac{1}{4}a^2-z^2}}{\frac{1}{2}a}$, and the whole surface is $S=4\displaystyle\int_0^{\frac{1}{2}a} adz=2a^2$ ....(1).

The center of gravity of S is distant from the base

$$z_1=\frac{4\int_0^{\frac{1}{2}a} azdz}{2a^2}=\tfrac{1}{4}a \text{ ....(2).}$$

For the volume, taking planes parallel to the base,

$$V=\int_0^{\frac{1}{2}a} 2\sqrt{\tfrac{1}{4}a^2-z^2}.2\sqrt{\tfrac{1}{4}a^2-z^2}.dz=\int_0^{\frac{1}{2}a}(a^2-4z^2)dz=a^3/3 \text{ ....(3),}$$

and its center of gravity above the base is

$$z_2=\frac{\int_0^{\frac{1}{2}a}(a^2-4z^2)zdz}{\frac{1}{3}a^3}=\tfrac{3}{16}a \text{ ....(4).}$$

The figure will be a cloistered arch formed by the intersection of two right semi-cylinders.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let $x^2+z^2=\frac{1}{4}a^2$......(1), $y^2+z^2=\frac{1}{4}a^2$......(2) be the equations to the cylinders which form the groin. From (1) $dz/dx=-x/z$, $dz/dy=0$.

$$S=\int\int\sqrt{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2}\,dxdy=8\int_0^{\frac{1}{2}a}\int_0^x\sqrt{1+\frac{x^2}{z^2}}\,dxdy$$

$$=4a\int_0^{\frac{1}{2}a}\int_0^x\frac{dxdy}{z}=4a\int_0^{\frac{1}{2}a}\int_0^x\frac{dxdy}{\sqrt{\frac{1}{4}a^2-x^2}}=4a\int_0^{\frac{1}{2}a}\frac{xdx}{\sqrt{\frac{1}{4}a^2-x^2}}=2a^2.$$

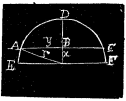
$$V=\int\int\int dzdxdy=4\int_0^{\frac{1}{2}a}\int_0^{\frac{1}{2}\sqrt{a^2-4z^2}}\int_0^{\frac{1}{2}\sqrt{a^2-4z^2}}dzdxdy=\int_0^{\frac{1}{2}a}(a^2-4z^2)dz=\tfrac{1}{3}a^3.$$

$$\text{Center of graving of surface}=\frac{\int\int zdS}{\int\int dS}=\tfrac{1}{2}a\int_0^{\frac{1}{2}a}\int_0^x dxdy=\tfrac{1}{4}a.$$

$$\text{Center of gravity of volume}=\frac{\int\int\int zdzdxdy}{\int\int\int dzdxdy}=\frac{3}{a^3}\int_0^{\frac{1}{2}a}z(a^2-4z^2)dz=\tfrac{3}{16}a.$$

IV. Solution by J. C. GREGG, A. M., Brazil, Indiana.

Let the given figure represent a section of the solid through the middle point of two opposite sides of the base. We have $r=a/2$, and the equation of the circle EDF is $x^2+y^2=r^2\ldots.(1)$, and $AC^2=(2y)^2=4(r^2-x^2)=A_x=$ a section parallel to the base, and for the volume



$$V=4\int_0^r(r^2-x^2)dx=\tfrac{8}{3}r^3=\tfrac{1}{3}a^3.$$

The surface may be considered to be generated by the sides of a section parallel to the base, and we have for the surface,

$$S=4\int 2yds=4\int_0^r 2\sqrt{r^2-x^2}.\frac{rdx}{\sqrt{r^2-x^2}},\ =8r\int_0^r dx=8r^2=2a^2.$$

For the center of gravity of the volume,

$$x=\frac{\int x(A_x)dx}{V},=\frac{4\int_0^r x(r^2-x^2)dx}{\frac{8}{3}r^3},=\frac{3}{2r^3}\int_0^r x(r^2-x^2)dx=\tfrac{3}{8}r=\tfrac{3}{16}a.$$

For the center of gravity of the *curved* surface we have,

$$\bar{x}=\frac{4\int 2xyds}{S},\ =\frac{8r\int_0^r xdx}{8r^2}=\frac{1}{r}\int_0^r xdx,\ =\tfrac{1}{2}r,\ =\tfrac{1}{4}a.$$

For the center of gravity of the *whole* surface, since the curved surface is *twice* that of the base we have, $\bar{x}=\frac{2}{3}.\frac{1}{4}a=\frac{1}{6}a$.

Also solved by *H. C. WHITAKER*, *C. W. M. BLACK*, and the *PROPOSER*.

Professors Black and Scheffer used "side$=2a$" as in Problem 47, instead of *side*$=a$, and hence their results did not agree with those in the published solutions. The results obtained were : Volume$=8a^3/3$, surface$=8a^2$, center of gravity of volume$=3a/8$, and center of gravity of surface$=\frac{1}{2}a$. See problem 42 for two additional SOLUTIONS for surface and volume. EDITOR.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

38. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

A prolate spheroid of revolution is fixed at its focus; a blow is given it at the extremity of the axis minor in a line tangent to the direction perpendicular to the axis major. Find the axis about which the body begins to rotate. [From *Loudon's Rigid Dynamics*.]

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

The general equations of motion are :

$$\left.\begin{array}{l} A\omega_x-(\Sigma mxy)\omega_y-(\Sigma mxz)\omega_z=L \\ B\omega_y-(\Sigma myz)\omega_z-(\Sigma myx)\omega_x=M \\ C\omega_z-(\Sigma mzx)\omega_x-(\Sigma mzy)\omega_y=N \end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The equation to the ellipsoid with focus as origin is $a^2y^2+a^2z^2+b^2x^2=2aeb^2x+b^4$. Now $\Sigma mxy=\Sigma mxz=\Sigma myz=0$. $\therefore$ (1) reduce to

$$\left.\begin{array}{l} A\omega_x=L \\ B\omega_y=M \\ C\omega_z=N \end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Let $2aeb^2x+b^4-b^2x^2=a^2c^2$. Then

$$A=\Sigma m(y^2+z^2)=\mu\int_{-a(1-e)}^{a(1+e)}\int_{-c}^{c}\int_{-\sqrt{c^2-y^2}}^{\sqrt{c^2-y^2}}(y^2+z^2)dxdydz$$

$$=\frac{4\mu}{3}\int_{-a(1-e)}^{a(1+e)}\int_{0}^{c}\{3y^2\sqrt{c^2-y^2}+(c^2-y^2)\sqrt{c^2-y^2}\}dxdy$$

$$=\frac{\pi\mu}{2a^4}\int_{-a(1-e)}^{a(1+e)}(2aeb^2x+b^4-b^2x^2)^2dx=\tfrac{8}{15}\mu\pi ab^4.$$

$$B=C=\Sigma m(x^2+y^2)=\mu\int_{-a(1-e)}^{a(1+e)}\int_{-c}^{c}\int_{-\sqrt{c^2-y^2}}^{\sqrt{c^2-y^2}}(x^2+y^2)dxdydz$$

$$=4\mu\int_{-a(1-e)}^{a(1+e)}\int_{0}^{c}(x^2+y^2)\sqrt{c^2-y^2}\,dxdy$$

$$=\frac{\mu\pi}{4}\int_{-a(1-e)}^{a(1+e)}(4c^2x^2+c^4)dx=\tfrac{8}{15}\mu\pi a^3b^2(1+2e^2).$$

Let the blow $=P$ be struck perpendicular to the plane (xy), then the moments of the impulsive forces about the axes are $L=Pb$, $M=Pae$, $N=0$.

These in (2) give

$$\left.\begin{array}{l}\tfrac{8}{15}\mu\pi ab^4\omega_x=Pb\\ \tfrac{8}{15}\mu\pi a^3b^2(1+2e^2)\omega_y=Pae\\ \omega_z=0\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

$$\therefore\ \frac{\omega_y}{\omega_x}=\frac{e^2}{1+2e^2}\cdot\frac{b}{ae}.$$

Let F be the focus, O the center of the ellipsoid. Then on the minor axis in the plane (xy), take $OE=\dfrac{e^2}{1+2e^2}.b$, then will FE be the axis required.

Let $a=5$, $b=4$. $\therefore e=\frac{3}{5}$. $\therefore OE=\frac{9}{43}b=\frac{36}{43}$. The resultant angular velocity will be

$$\frac{15P}{8\mu\pi a^2b^3e}.OF=\frac{P}{512\mu\pi}.OF=\frac{3P\sqrt{1993}}{22016\mu\pi},\text{ when } a=5,\ b=4.$$

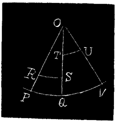
39. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

A person whose height is a and weight W stands in a swing whose length is l. Supposing the initial inclination of the swing to the vertical is α and that the person always crouches when in the highest position and stands up when in the lowest, his center of gravity moving through a distance b measured from lower part of swing upward, find how much the arc is increased after n complete vibrations.

I. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

Let RS be the path of the center of gravity from extreme position to vertical, TU the path from vertical to other extreme position.

$OP=l$, $RP=k$, (say), $TS=b$; $\angle ROS=a$, $\angle TOU=\alpha_1$, etc.



The energy acquired by swing in passing from R to S is $(l-k)(1-\cos\alpha)W$.

When it has passed to V the energy is $(l-k-b)(1-\cos\alpha_1)W$.

By conservation of energy

$$(l-k-b)(1-\cos\alpha_1)W=(l-k)(1-\cos\alpha)W.$$

$$\text{Whence } 1-\cos\alpha_1=\frac{l-k}{l-k-b}(1-\cos\alpha).$$

In passing back to original position

$$1-\cos\alpha_2=\frac{l-k}{l-k-b}(1-\cos\alpha_1)=\left(\frac{l-k}{l-k-b}\right)^2(1-\cos\alpha).$$

$$\text{For two complete vibrations } 1-\cos\alpha_4=\left(\frac{l-k}{l-k-b}\right)^4(1-\cos\alpha).$$

In like manner for n complete vibrations

$$1-\cos\alpha_{2n}=\left(\frac{l-k}{l-k-b}\right)^{2n}(1-\cos\alpha) \text{ or } \sin(\tfrac{1}{2}\alpha_{2n})=\left(\frac{l-k}{l-k-b}\right)^n\sin(\tfrac{1}{2}\alpha),$$

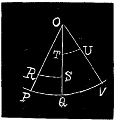
which enables us to compute the increase in amplitude.

II. Solution by the PROPOSER.

Let O be the point of suspension of the swing, S the position of the center of gravity of the man when crouching, and T its position when the man is standing, and Q the lower end of the swing.

Let $OQ=l$, $SQ=k$, the distance from lower end of swing to the center of

gravity of the man when he is crouching, $ST=b$, $\angle QOP=\alpha$, and $\angle QOV=\theta_0$.



Then the velocity, v, of the man at the point Q, $=\sqrt{2gl(l-k)(1-\cos\alpha)}$, *Bowser's Analytical Mechanics*, page 350, Article 192. Hence, his energy due to his weight $=\frac{W}{2g}v^2=W(l-k)(1-\cos\alpha)$. This energy will carry him to V and equals $W(l-k-b)(1-\cos\theta_0)$, since the man rises at the point Q. Since he crouches at the point V, his energy at the point Q on his return is $W(l-k)(1-\cos\theta_0)$. This energy will carry him to a point to the left of S, and the energy expended will be $W(l-k-b)(1-\cos\theta_1)$, where θ_1 is the angle between the vertical and the swing.

According to the principle of the conservation of energy, we have,

$$W(l-k)(1-\cos\alpha)=W(l-k-b)(1-\cos\theta_1)\ldots\ldots\ldots\ldots\ldots(1),$$

$$W(l-k)(1-\cos\theta_0)=W(l-k-b)(1-\cos\theta_2)\ldots\ldots\ldots\ldots\ldots(2),$$

$$W(l-k)(1-\cos\theta_1)=W(l-k-b)(1-\cos\theta_3)\ldots\ldots\ldots\ldots\ldots(3),$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$W(l-k)(1-\cos\theta_{2n-1})=W(l-k-b)(1-\cos\theta_{2n-1})\ldots\ldots(2n).$$

Multiplying these equations together member for member, and solving for $1-\cos\theta_{2n}$, we have,

$$1-\cos\theta_{2n}=\left(\frac{l-k}{l-k-b}\right)^{2n}(1-\cos\alpha),\text{ or }\sin^2(\tfrac{1}{2}\theta_{2n})=\left(\frac{l-k}{l-k-b}\right)^{2n}(\sin^2\tfrac{1}{2}\alpha).$$

$$\text{Whence, }\sin\tfrac{1}{2}\theta_{2n}=\left(\frac{l-k}{l-k-b}\right)^{n}\sin\tfrac{1}{2}\alpha,\text{ or }\theta_{2n}=2\sin^{-1}\left[\left(\frac{l-k}{l-k-b}\right)\sin\tfrac{1}{2}\alpha\right].$$

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

42. Proposed by E. B. ESCOTT, Chicago, Illinois.

Required all the parallelograms whose sides a, b and diagonals c, d, are rational.

II. Solution by W. F. KING, Ottowa, Canada.

The conditions are, $2(a^2+b^2)=c^2+d^2$, with the condition that a, b, c and a, b, d shall be capable of forming triangles (sum of any two sides greater than the third side). That is, if we suppose $a>b$, $c>d$, $a+b$ must be greater than c, and $a-b<d$. These two conditions again are the same, for if $a+b>c$ and $2(a^2+b^2)=c^2+d^2$, $2(a^2+b^2)-(a+b)^2<d^2$ or $a-b<d$.

Let us suppose the numbers involved to be integers. We have $c^2+d^2=2(a^2+b^2)=(a+b)^2+(a-b)^2$. If $c=a+b$, $d=a-b$, the parallelogram vanishes, the angle opposite to the diagonal c becoming 180°. But if not, we have a number c^2+d^2 which is resolvable into the sum of two squares in another way. Hence, as may easily be proved, a^2+b^2 is resolvable into factors, which, by a theorem in the Theory of Numbers, are (whether prime or composite) each expressible as the sum of two squares. Also every prime number of the form $4n+1$ is expressible as the sum of two squares, and every number which is the sum of two squares is the product of prime factors of the form $4n+1$. (The only even prime $2=1^2+1^2$ or a power thereof may also be a factor, which case will be considered further on).

Hence we have a rule to find a and b so as to make c and d rational.

Form a^2+b^2 by multiplying together two or more of the various prime numbers of the form $4n+1$, such as 5, 13, 17, 29, etc.

The product may be expressed in two ways at least as the sum of two squares. Thus we shall have $f^2+g^2=h^2+k^2$.

$\therefore$ $2(f^2+g^2)=(h^2+k)^2+(h-k)^2$, which gives a solution by putting $f=a$, $g=b$, $h+k=c$, $h-k=d$, provided that, (following the condition for a possible triangle) $f-g<h-k$.

If $f-g>h-k$, we must take h and k for a and b ; $f+g$ and $f-g$ for c and d. Then $2(h^2+k^2)=(f+g)^2+(f-g)^2$.

That is, of the two equal sums into which the product has been resolved, take that for a^2+b^2 which has the less difference between its components.

For example, multiply 5 by $13=65$.

$65=8^2+1^2=7^2+4^2$ and $7-4=3<8-1$. Hence $a=7$, $b=4$, $h=8$, $k=1$, and $2(7^2+4^2)=(8+1)^2+(8-1)^2=9^2+7^2$.

$\therefore$ $c=9$, $d=7$. And 7, 4, 9 ; 7, 4, 7 are possible sides for triangles. The components of the product can readily be found from the components of the prime factors, thus :

Let $N=(p^2+q^2)(r^2+s^2)=p^2r^2+2pqrs+q^2s^2+q^2r^2-2pqrs+p^2s^2=p^2r^2-2pqrs+q^2s^2+q^2r^2+2pqrs+p^2s^2=(pr+qs)^2(qr-ps)^2=(pr-qs)^2+(qr+ps)^2$.

For example, to resolve 65 into the sum of two squares.

$65=13\times5=(3^2+2^2)(2^2+1^2)$. Here $pr+qs=3\times2+2\times1=8$.

$qr-ps=2\times2-3\times1=1$. $pr-qs=3\times2-2\times1=4$,

$qr+ps=2\times2+3\times1=7$. $\therefore$ $65=8^2+1^2=4^2+7^2$.

A third factor, $r_1^2+s_1^2$, can be introduced by putting $pr+qs=p_1$, $qr-ps=q_1$, and multiplying out $(p_1^2+q_1^2)(r_1^2+s_1^2)$ as before.

Observe that this gives two forms for the product, and two more would be got by putting $pr-qs=p_1$, $qr+ps=q_1$, so that with three factors there will be four forms for the product. These forms may be taken any two together giving $4.3/1.2=6$ solutions for c and d.

In the preceding it has been assumed that a and b have no common factor. If they have one (which may be any number whatever) the preceding investigation will still hold. But such factors of the common measure of a and b as are not primes of the form $4n+1$ will re-appear as common factors of c and d. It is to be noted that if $a^2+b^2=2\times$ a single prime factor a^2+b^2 can be expressed as the sum of two squares in one way only, viz: $a^2+b^2=2(p^2+q^2)=(p+q)^2+(p-q)^2$. For the factors of a^2+b^2 are p^2+q^2 and r^2+s^2 where $r=s=1$, and in multiplying $(p^2+q^2)(r^2+s^2)$ the two expressions $(qr+ps)^2+(pr-qs)^2$ and $(pr+qs)^2+(qr-ps)^2$ become identical when $r=s=1$, and these two cannot be equated to a third and different sum of two squares without factoring p^2+q^2, which is by supposition a prime. Hence when $\frac{1}{2}(a^2+b^2)$ is a prime the solution fails, for we get $2(a^2+b^2)=4(p^2+q^2)=(2p)^2+(2q)^2=(a+b)^2+(a-b)^2$, which does not give a parallelogram. So also when a^2+b^2 is a product of a prime by any odd power of 2. An even power of 2 may however be used, for example,

$$a^2+b^2=260=2^2\times5\times13=2^2(3^2+2^2)(2^2+1^2)=(6^2+4^2)(2^2+1^2)=16^2+2^2$$
$$=14^2+8^2 \text{ and } 2(a^2+b^2)=2(14^2+8^2)=2(16^2+2^2)=18^2+14^2=c^2+d^2.$$

The above discussion made on the assumption that a, b, c, d are integers, is readily extended to give solutions in rational fractions.

Thus $1885=5\times13\times29=42^2+11^2=34^2+27^2$.

$\therefore\ 7^2+(\frac{11}{6})^2=(\frac{17}{3})^2+(\frac{9}{2})^2$ and $2\{(\frac{17}{3})^2+(\frac{9}{2})^2\}=(\frac{53}{6})^2+(\frac{31}{6})^2$.

Note on solution of problem 37, page 151. The failure to give the least values in my solution was due to solving $x_1{}^2-40y_1{}^2=1$; by continued fractions, we obtain positive integral values for x and y, but y does not enter into the required values directly; hence may be fractional. This point was overlooked.

To obtain all these values, let $x=(x_2)\div(z_2)$, $y=(y_2)\div(z_2)$, and then $x_2{}^2-z_2{}^2=40y_2{}^2=10y_3{}^2$. $(x_2+z_2)(x_2-z_2)=10y_3{}^2$. Let $y_3{}^2=p^2q^2$ and $10=$ any two factors.

$$\left.\begin{array}{l} x_2+z_2=p^2 \text{ or } 2p^2 \\ x_2-z_2=10q^2 \text{ or } 5q^2 \end{array}\right\}$$

Add and subtract, then

$$\left.\begin{array}{l} x_2=p^2+10q^2 \text{ or } 2p^2+5q^2 \\ z_2=\mp p^2\pm10q^2 \text{ or } \mp2p^2\pm5q^2 \\ y_3=2pq \end{array}\right\}$$

p and q taken at pleasure will give an infinite number of values, integral and fractional. Mr. Gruber's list is correct.

A. H. Bell.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

Note on Problem 33. By Henry Heaton, Atlantic, Iowa.

The result obtained in the published solution of this problem cannot be correct.

The area of the regular pentagon is $3.6327a^2$. The area of each of the infinite number of regular polygons whose apothem is a and number of sides greater than five, is less than $3.6327a^2$, while that of only two, the square and triangle, is greater. Hence the average area of all regular polygons with apothem a is less than $3.6327a^2$. Hence the result obtained in the published solution ($3.8693a^2$) is too large.

In a similar manner it may be shown that any result larger than $a^2\pi$ is too large, while it is evident that any result smaller than that is too small.

37. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

Required the average area of all triangles two of whose sides are a and b.

I. Solution by the PROPOSER.

It is well known that every triangle consists of six parts, three sides and three angles, and one side with any two other parts determines the triangle.

In constructing this triangle we may use all possible values, first, of the included angle C, second, of the third side, c, third, of the angle A, and fourth, of the angle B. This gives us four cases.

I. Put angle $C=\theta$. Then $A_1=\dfrac{ab}{2}\displaystyle\int_0^\pi \sin\theta d\theta \div \int_0^\pi d\theta=\frac{ab}{\pi}$.

II. Put side $c=x$. Then $A_2=\frac{1}{4}\displaystyle\int_{a-b}^{a+b}[(a+b)^2-x^2]^{\frac{1}{2}}[x^2-(a-b)^2]^{\frac{1}{2}}dx$.

$$\div\int_{a-b}^{a+b}dx=\frac{a+b}{12b}\left\{(a^2+b^2)E\left[\left(\frac{2\sqrt{ab}}{a-b}\right),\ \tfrac{1}{2}\pi\right]-(a-b)^2F\left[\left(\frac{2\sqrt{ab}}{a+b}\right),\ \tfrac{1}{2}\pi\right]\right\}.$$

(To integrate this expression put $x=[(a+b)^2-4ab\sin\theta]^{\frac{1}{2}}$).

III. Put angle $A=\theta$, b being $<a$, then

$$A_3=\tfrac{1}{2}b\int_0^\pi[b\cos\theta+(a^2-b^2\sin^2\theta)^{\frac{1}{2}}]\sin\theta d\theta\div\int_0^\pi d\theta=\frac{ab}{2\pi}+\left(\frac{a^2-b^2}{4\pi}\right)\log_e\left(\frac{a+b}{a-b}\right).$$

IV. Put angle $B=\theta$. For every value of B there are two triangles whose average arc is $\frac{1}{2}a^2\sin\theta\cos\theta$. Hence,

$$A_4=\tfrac{1}{2}a^2\int_0^{\sin^{-1}\frac{b}{a}}\sin\theta\cos\theta d\theta\div\int_0^{\sin^{-1}\frac{b}{a}}d\theta=\tfrac{1}{4}b^2\div\sin^{-1}\frac{b}{a}.$$

COROLLARY. If $b=a$, $A_1=a^2/\pi$, and $A^2=a^2/3$. These are double the values found in the solutions of problem 26, as they evidently should be. The values of A_3 and A_4 do not hold when $b=a$, for the reason that while the sum of the areas remains the same the number of triangles is reduced one-half at the moment that $b=a$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let x=third side, A=area, Δ=average area.

$$\therefore A=\tfrac{1}{4}\sqrt{(a+b)^2-x^2}\sqrt{x^2-(a-b)^2}.$$

$$\Delta=\int_{a-b}^{a+b}A\,dx\div\int_{a-b}^{a+b}dx=\frac{1}{2b}\int_{a-b}^{a+b}A\,dx.$$

Let $(a+b)^2-x^2=4aby^2$, $\dfrac{4ab}{(a+b)^2}=e^2$.

$$\therefore \Delta=\frac{2a^2b}{a+b}\int_0^1\frac{y^2\sqrt{1-y^2}\,dy}{\sqrt{1-e^2y^2}}=\tfrac{1}{2}a(a+b)\int_0^1\frac{y^2\sqrt{1-e^2y^2}\,dy}{\sqrt{1-y^2}}$$

$$-\frac{a(a-b)^2}{2(a+b)}\int_0^1\frac{y^2\,dy}{\sqrt{1-y^2}\sqrt{1-e^2y^2}}=\frac{a+b}{12b}\{(a^2+b^2)E(e)-(a-b)^2F(e)\}.$$

III. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C., and A. P. REED, Clarence, Missouri.

The area of triangle is $\triangle=\frac{1}{2}ab\sin\theta$.

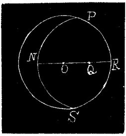
$$\text{Hence, average area}=\tfrac{1}{2}ab\int_0^\pi\sin\theta d\theta\div\int_0^\pi d\theta=\tfrac{1}{2}ab\Big[-\cos\theta\Big]_0^\pi\div\pi=\frac{ab}{\pi}.$$

38. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

Two arrows are sticking in a circular target : show that the chance that their distance is greater than the radius of the target is $3\sqrt{3}/4\pi$. [From *Todhunter's Integral Calculus*, page 335.]

I. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

Let Q be the position of one arrow. Call the radius of target R, and let $OQ=\rho$.



The area $PNSR=2R^2\cos^{-1}\frac{\rho}{2R}-\rho\sqrt{R^2-(\frac{1}{2}\rho)^2}$.

Then the chance that the second arrow is within the above region is

$$\frac{2}{\pi}\cos^{-1}\frac{\rho}{2R}-\frac{\rho}{\pi R^2}\sqrt{R^2-(\tfrac{1}{2}\rho)^2}.$$

The chance that the first arrow is at a distance ρ from the center is

$$\frac{2\pi\rho d\rho}{\pi R^2}=\frac{2\rho d\rho}{R^2}.$$

The chance that the *two* arrows are as indicated above is

$$\frac{4}{\pi R^2}\cos^{-1}\frac{\rho}{2R}\cdot\rho d\rho-\frac{2}{\pi R^4}\sqrt{R^2-(\tfrac{1}{2}\rho)^2}\,\rho^2 d\rho.$$

The sum of all such chances is

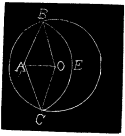
$$\frac{4}{\pi R^2}\int_0^R\rho\cos^{-1}\frac{\rho}{2R}d\rho-\frac{2}{\pi R^4}\int_0^R\sqrt{R^2-(\tfrac{1}{2}\rho)^2}\,\rho^2 d\rho=1-\frac{3\sqrt{3}}{4\pi}.$$

$\therefore$ Chance that the second arrow is *without* the region $PNSR$ is

$$1-\left(1-\frac{3\sqrt{3}}{4\pi}\right)=\frac{3\sqrt{3}}{4\pi}.$$

II. Solution by HENRY HEATON, M. Sc., Atlantic, Iowa.

Let O be the center of the target and A the position of one of the arrows. Then if the distance between the arrows is greater than the radius, a, the other arrow must lie outside the arc BEC drawn from A as center and radius a. If $AO=x$, the area of the surface outside of the arc BEC is



$$S=2a^2\sin^{-1}\left(\frac{x}{2a}\right)+\tfrac{1}{2}x(4a^2-x^2)^{\frac{1}{2}}.$$

The probability that the one arrow is at the distance x from the center is $2\pi xdx\div a^2\pi=2xdx\div a^2$. The probability that the

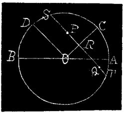
other is on the surface outside the arc BEC is $S\div a^2\pi$. Hence the required probability is

$$P=\frac{2}{a^4\pi}\int_0^a Sxdx. \quad \text{Put } x=2a\sin\theta.$$

$$\text{Then } P=\frac{16}{\pi}\int_0^{\frac{1}{6}\pi}(\theta+\sin\theta\cos\theta)\sin\theta\cos\theta=3\sqrt{3}/4\pi.$$

III. **Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.**

Let P, Q be the arrows, $SQ=x$, $PQ=y$, $ST=u$, $OR=z$, $\angle DOB=\theta$, $OA=a$.



An element of area at Q is $dzdx$; at P, $yd\theta dy$.

The limits of x and 0 are $u-a$; of y, $u-x$ and a, and doubled ; of z, 0 and $\frac{1}{2}a\sqrt{3}$, and doubled ; of θ, 0 and $\frac{1}{2}\pi$, and doubled. Δ=chance, $u=2\sqrt{a^2-z^2}$.

$$\therefore \Delta=\frac{8}{\pi^2a^4}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}a\sqrt{3}}\int_0^{u-a}\int_a^{u-x}d\theta dzydxdy$$

$$=\frac{4}{\pi^2a^4}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}a\sqrt{3}}\int_0^{u-a}\{(u-x)^2-a^2\}d\theta dzdx$$

$$=\frac{4}{3\pi^2a^4}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}a\sqrt{3}}(u^3+2a^3-3a^2u)d\theta dz=\frac{3\sqrt{3}}{2\pi^2}\int_0^{\frac{1}{2}\pi}d\theta=\frac{3\sqrt{3}}{4\pi}.$$

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

34. **Proposed by THOS. U. TAYLOR, C. E., M. C.; Department of Engineering, University of Texas, Austin, Texas.**

Given a variable parallelogram $ABCP$, where P remains fixed. A moves on an irregular plane curve (closed) and C moves on another irregular plane curve (closed) whose plane is parallel to the plane of (A) curve. The generator PC moves completely around and returns to its initial position, AB always moving parallel to PC, and, of course, returns to its initial position. If distance between planes (A) and (C)$=h$, show by elementary mathematics and without using theorem of Koppe that volume of solid generated by variable parallelogram $ABCP=\frac{1}{2}h$ (area generated by AP+area generated by BC).

Solution by the PROPOSER.

Let (A)=area generated by PA ; (B)=area curve generated by B ; (C) =area curve generated by C.

Project area (A) orthogonally on plane of (B) and (C). Then by Elliott's Extension of Holditch's Theorem

$$S=x(A)+y(B)-xy(C).$$

where $x+y=1$, and x and y are the radii in which the section S divides the generator. Make $x=y=\frac{1}{2}$.

$$\therefore\ S_{\frac{1}{2}}=\tfrac{1}{2}(A)+\tfrac{1}{2}(B)-\tfrac{1}{4}(C).$$

But by Newton's formula, V=volume of whole solid

$$=\tfrac{1}{6}H\{(A)+4S_{\frac{1}{2}}+(B)\}=H\{\tfrac{1}{2}[(B)+(A)]-\tfrac{1}{6}(C)\}.$$

Volume of cone$=\frac{1}{3}H(C)$. $\therefore$ Volume generated by

$$APCB=\tfrac{1}{2}H\{(A)+[(B)-(C)]\}=\tfrac{1}{2}H\{\text{area } AP+\text{area } BC\}.$$

34. Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy in Pacific College, Santa Rosa, California ; P. O., Sebastopol, California.

To an observer whose latitude is 40 degrees north, what is the sidereal time when Fomalhout and Antares have the same altitude: taking the Right Ascension and Declination of the former to be 22 hours, 52 minntes, —30 degrees, 12 minutes; of the latter, 16 hours, 23 minutes, —26 degrees, 12 minutes ?

II. Solution (continued) by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

$h=130^\circ\ 4'\ 57''$ for upper meridian.

$\therefore\ 180^\circ-130^\circ\ 4'\ 57''=49^\circ\ 55'\ 3''=h$, for lower meridian.

$\therefore\ h=3$ hours, 19 minutes, 40.2 seconds.

$\therefore$ sidereal time for equal altitudes on latitude 40° south is $a-h=19$ hours, 32 minutes, 19.8 seconds.

$a-h-12=7$ hours, 32 minutes, 19.8 seconds is sidereal time on upper meridian at same moment.

Dr. S. Hart Wright communicated to me the following rather startling discovery which is probably responsible for the problem : The arc of a great circle passing to and between the two stars actually passed through the Nadir. Now when the stars are of equal altitudes they are equally distant from the Nadir as well as from the Zenith.

$\therefore$ The arc between them$=82^\circ\ 51'\ 52.5''$ must be bisected, each being $41^\circ\ 25'\ 56\frac{1}{4}''$.

These facts, if they had been stated, would have made the problem quite simple.

See problem and solution in August-September number.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

71. Proposed by J. C. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

A man owes me \$200 due in 2 years, and I owe him \$100 due in 4 years; when can he pay me \$100 to settle the account equitably, money being worth 6% ?

72. Proposed by W. H. CARTER, Professor of Mathematics, Centenary College of Louisiana, Jackson, Louisiana.

Though the length of my field is 1-7 longer than my neighbor's, and its quality is 1-9 better, yet as its breadth is 1-4 less, his is worth \$500 more than mine. What is mine worth? *Encyclopedia Brittanica.*

73. Proposed by NELSON S. RORAY, South Jersey Institute, Bridgeton, New Jersey.

I would like to change problem 70, Arithmetic, to read as follows and have it proposed for solution:

A owes me \$100 due in 2 years, and I owe him \$200 due in 4 years. When can I pay him \$100 to settle the account equitably, money being worth 6%, and *the interest to draw interest until the time of settlement ?*

Solve by simple arithmetic without the aid of algebraic symbols.

74. Proposed by JOHN T. FAIRCHILD, Principal of Crawfis College, Crawfis College, Ohio.

When U. S. bonds are quoted in London at $108\frac{3}{4}$ and in Philadelphia at $112\frac{1}{4}$, exchange $\$4.89\frac{1}{2}$, gold quoted at 107, how much more was a \$1000 U. S. bond worth in London than in Philadelphia?

ALGEBRA.

74. Proposed by NELSON S. RORAY, South Jersey Institute, Bridgeton, New Jersey.

Solve according to the conditions given:

$$\sqrt{x+1}+\sqrt{x}=\frac{3}{\sqrt{1+x}}.$$

First, square without transposing and then solve; second, transpose $\sqrt{x+1}$ and then solve. Obtain the same roots as in the first way of solving.

75. Proposed by B. F. BURLESON, Oneida Castle, New York.

Mr. B's farm is in shape a quadrilateral, both inscriptible and circumscriptible, and contains an area of $k=10752$ square rods. The square described on the radius of its inscribed circle contains $r^2=2304$ square rods; while the square described on the radius of its circumscribed circle contains an area of $R^2=7345$ square rods. Required the lengths of the sides of his farm.

76. Proposed by E. B. ESCOTT, Fellow in Mathematics, University of Chicago, Chicago, Illinois.

Prove the identities

$$2-\sqrt{2}=\tfrac{1}{2}+\frac{1}{2^2.3}+\frac{1}{2^3.3.17}+\frac{1}{2^4.3.17.577}\cdots\cdots$$

$$\frac{5-\sqrt{5}}{2}=\tfrac{1}{1}+\tfrac{1}{3}+\frac{1}{3.7}+\frac{1}{3.7.47}+\frac{1}{3.7.47.2207}\cdots\cdots$$

77. Proposed by G. I. HOPKINS, Instructor in Mathematics and Physics in High School, Manchester, New Hampshire.

Solve the equation, $(6x^2+x-3)^2-48^2=(x+15)^2$.

GEOMETRY.

69. Proposed by WILLIAM SYMMONDS, M. A., Professor of Mathematics and Astronomy in Pacific College, Santa Rosa, California; P. O., Sebastopol, California.

To divide a square card into right-lined sections in a manner, that a rectangle of a given breadth can be formed from the sections; likewise, form a square from a rectangular card.

70. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, Ohio.

Prove that the locus of the center of the circle which passes through the vertex of a parabola and through its intersections with a normal chord is the parabola $2y^2=ax-a^2$, the equation to the given parabola being $y^2=4ax$.

71. Prove by pure geometry: A perpendicular at the middle point, M_a, of the side BC of the triangle ABC meets the circumcircle in A'. On this perpendicular A'' and A''' are taken so that $M_aA''=M_aA'$ and $A''A'''=AH$. (H is the orthocenter of triangle ABC). Prove that A''' is on the circumcircle. *Anon.*

72. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

If a line with its extremities upon two curves move in any manner whatever, (the line may vary in length), and P a point upon the line which divides it in the ratio $m:n$ describe a curve, the area of this curve will be given by the formula—

$$A=\frac{(m^2+nm)A_1+(n^2+mn)A_2-mnA_3}{(m+n)^2}.$$

73. Prove by pure geometry: (1) A', B', and C' are the middle points of the arcs BC, CA, and AB respectively. With these points as centers, circles are described passing through B and C, C and A, and A and B respectively. Prove that these circles intersect in O, the center of the incircle of the triangle ABC; (2), that O, the center of the incircle, is Nagel's point of the triangle formed by joining the middle points of the sides. *Anonymous.*

CALCULUS.

61. Proposed by W. H. CARTER, Professor of Mathematics, Centenary College of Louisiana, Jackson, Louisiana.

If $r=a\sin n\theta$ is the polar equation of a curve, show (1) that the curve consists of n or $2n$ loops according as n is an odd or an even integer; (2) that its area is $\frac{1}{4}$ or $\frac{1}{2}$ of the circumscribing circle according as n is an add or an even integer.

62. Proposed by A. H. HOLMES, Brunswick, Maine.

A bucket is in the form of a frustum of a cone having its smaller end as a base. It is a inches in diameter at base and b inches in diameter at top, and its perpendicular

height is c inches. It contains water the perpendicular height of which is $\frac{1}{2}c$ inches. What is the greatest height, from the plane on which the vessel rests, to which the surface of the water will rise when the bucket is overturned, no allowance being made for the thickness of the material of the bucket.

63. **Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.**

What is the volume removed by boring an auger hole radius R through a right cylinder radius R, the center of the auger hole to pass at a distance c from the axis of the cylinder and inclined to the axis at an angle α ?

64. **Proposed by E. S. LOOMIS, A. M., Ph. D., Professor of Mathematics, High School, Cleveland, Ohio.**

Find volume and surface generated by revolving about y, the catenary $y=\frac{1}{2}a(e^{x\backslash a}+e^{-x\backslash a})$, from $x=0$ to $x=a$. [*Osborne's Calculus*, page 255, example 8.]

MECHANICS.

48. **Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas, Texas.**

Two equal heavy rings connected by a string passing over a peg at the focus of a conic section will be in equilibrium at all points on the curve.

49. **Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.**

A rectangular stick of timber of known dimensions is placed upon a platform of given height in a vertical position with the center above the edge of platform, and slightly displaced from the vertical. Where and in what manner will it strike the ground.

50. **Proposed by J. SCHEFFER, A. M., Hagerstown, Maryland.**

A plane quadrilateral $ABCD$ in the vertical wall of a cistern, filled with water, has its four vertices A, B, C, D at the distances 10 feet, 4 feet, 5 feet, and 7 feet respectively, from the surface of the water. The projections of AB, BC, and CD upon the surface are respectively 2 feet, 3 feet, and 1 foot. Find the pressure of the water upon the quadrilateral, and the position of the center of mean pressure.

51. **Proposed by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.**

"Swift of foot was Hiawatha.
He could shoot an arrow from him
And run forward with such fleetness
That the arrow fell behind him!
Strong of arm was Hiawatha;
He could shoot ten arrows upward
Shoot them with such strength and swiftness
That the tenth had left the bowstring
Ere the first to earth had fallen." *Longfellow.*

Assuming Hiawatha to have been able to shoot an arrow every second and to have aimed when not shooting vertically so that the arrow might have the longest range; what was Hiawatha's time in a hundred yards?

AVERAGE AND PROBABILITY.

47. **Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.**

What is the average length of the chords that may be drawn from one extremity of the major axis of an ellipse to every point of the curve ?

48. **Proposed by P. H. PHILBRICK, C. E., Pineville, Louisiana.**

A, B, C, D, and E play with dice, each throwing three, three successive times, for a stake a. A, B, and C throw; C throwing the highest, 52. What is his expectation ?

49. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

A square whose side is $2a$ and an equilateral triangle whose altitude is $3a$ are fastened together at their centers, but otherwise free to move. If they are thrown on a floor at random, what is the average area common to both?

50. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Find (1), the average length of all straight lines having a given direction, between 0 and a; (2), the average length of chords drawn from one extremity of the diameter a of a semi-circle to all points in the semi-circumference; and (3), find the average area of all triangles formed by a straight line of constant length a sliding between two straight lines at right angles.

[Solutions of these problems should be sent to the editors of the respective departments on or before February 1, 1897.]

EDITORIALS.

Our valued contributor, Prof. O. W. Anthony, has been elected Professor of Mathematics in the Columbian University, Washington, D. C.

James F. Lawrence, I. F. Yothers, G. B. M. Zerr, J. C. Corbin, Frederick R. Honey, H. C. Wilkes, and Nelson S. Roray should have received credit for solving Nos. 66, 67, 68, and 69, Department of Arithmetic. O. W. Anthony should have received credit for solving No. 63, Department of Geometry. We wish to state again that all solutions, to receive credit, should be sent to the proper editor; but this remark does not apply to the above persons.

The MONTHLY will soon begin its fourth volume. Will not every one of its old subscribers try and secure one new subscriber for the coming year? Send us names of persons likely to subscribe and we shall take pleasure in sending them sample copies. Persons sending us three new subscribers and remitting us $6.00 will receive a years subscription as a premium.

Some of our readers have suggested that we publish in groups portraits of our contributors. If this suggestion meets the approval of our contributors, we shall be pleased to receive photos which we will have grouped by one of the best artists in Springfield, and shall furnish the plates at cost to us. We shall be pleased to hear from the contributors to the MONTHLY in reference to this matter.

A letter from Dr. Halsted dated November 27th, 1896, says, "For four months I was buried in the uttermost parts of Hungary, Russia, and Siberia, and am just getting used to English again. I made many important finds and had many strange experiences." There are few other Americans whose travels in Russia would have been as important to the Non-Euclidean Geometry as this trip of Dr. Halsted's. He is already working on some very important translations which will soon be made known for the first time to English speaking mathematicians.

BOOKS AND PERIODICALS.

Elements of Mechanics, Including Kenematics, Kinetics, and Statics, with Applications. By Thomas Wallace Wright, M. A., Ph. D., Professor in Union College. 8vo. Cloth, 372 pages. Price, $2.50. New York: D. Van Nostrand Company.

This is a completely rewritten edition of the author's Text-book of Mechanics. The same general plan has been followed, but many changes in detail have made, so the book comes before the public with a new name. In this book much use is made of the graphical method; machines are discussed in great detail; the important subjects of oscillation and rotation have been treated with more fullness than is usual in an elementary treatise. Numerous well chosen problems are appended to the discussion, while at the end of each chapter is added a series of examination questions. Historical notes are freely interspersed to add a more live interest to the subject. This is a very excellent book and we very heartily recommend it to teachers desiring a good work on Mechanics. B. F. F.

The Elements of Physics. A College Text-book. By Edward L. Nichols and William S. Franklin. In three volumes, Vol. II. Electricity and Magnetism. 8vo. Cloth, ix and 272 pages. Price, $1.50. New York: The Macmillan Co.

In the study of this excellent work a knowledge of the elementary principles of the calculus and quaternions is required. This fact will preclude its use in many colleges. The authors recognizing, however, that there is a growing tendency among the best colleges to increase the requirements in mathematics, these colleges realizing that the discipline received from the study of mathematics is not excelled by any other branch of study, have not slurred over certain parts of Physics containing *real and unavoidable difficulties*. Nor have those portions containing these difficulties been omitted, but they have been faced frankly; the statements involving them having been reduced to the simplest form which is compatible with accuracy. Colleges in which only one course is offered in Physics should at once so adjust their courses of study as to make it possible to use a text-book such as the one before us, as a course of Physics pursued in accordance with the plan of this work will be of infinitely more value both from a practical and an educational point of view, than two or three popular courses requiring only a knowledge of Elementary Algebra and Geometry. B. F. F.

Elements of Plane and Spherical Trigonometry. By C. W. Crockett, Professor of Mathematics and Astronomy, Renssellaer Polytechnic Institute, Troy, New York. Large 8vo. Cloth, 192 pages and 120 pages of tables. Price, $1.25. New York and Chicago: American Book Company.

This work is fully up to the standard of good text-books. It contains a full course in Plane and Spherical Trigonometry; in fact, all that is needed in a course in the best schools and colleges. There are many examples and illustrations. The typographical and mechanical execution of the work is first-class. B. F. F.

Darwinism and Non-Euclidean Geometry. Reprint from the Bulletin de La Société Physico-Mathématique de Kasan. Tome VI. No. 3—4. By Dr. George Bruce Halsted. Pamphlet, 4 pages.

This interesting article seems to have been written by Dr. Halsted while visiting at Kasan in July and August of last summer. In his travels he explored many libraries and made many important finds. B. F. F.

The Maine Farmer's Almanac for 1897.

Through the courtesy of Prof. William Hoover, of Athens, Ohio, we received a copy of this noted little Almanac, which, among other important and useful information, contains two pages devoted to Mathematical Questions and Solutions. The price of the Almanac is 10 cents. B. F. F.

Prismoidal Formulae, with Special Derivation of Two-Term Formulae. By Thomas U. Taylor, C. E. (University of Virginia), M. C. E. (Cornell), Associate Professor of Applied Mathematics, University of Texas. Pamphlet, 55 pages.

This paper, which was read before the Texas Academy of Science, March, 1896, adds some valuable material to the literature of Prismoidal Formulae. B. F. F.

Mathematical Questions and Solutions. From the "Educational Times," with an Appendix. Edited by W. J. C. Miller, B. A. Vol. LXV., 8vo. Boards, 128 pages. Francis Hodgson, 89 Farringdon Street, E. C., London.

This valuable reprint contains solutions of about 165 problems. Our readers who secure it will find many interesting problems with their solutions. The price is 5s., 3d., postpaid. J. M. C.

Elementary Hydro-Statics. University Tutorial Series. By William Briggs and G. H. Bryan. Cloth, 208 pages. Price, 50 cents. New York: W. B. Clive, 65 Fifth Avenue.

This work is written in a suggestive and attractive manner. In scope and in method it is admirably adapted to class use as an elementary text. In the examples results are deduced from first principles, and thus the student is not lead to rely on memory for his formulae. The new features are good, the examples are numerous and well selected, and the topical index convenient and useful. J. M. C.

Inductive Manual of Straight Line and Circle. By William J. Meyers, Professor of Mathematics, State Agricultural College of Colorado. Published by the Author, Fort Collins, Colorado, 1896. 113 pages. Price, 60 cents.

The fundamental idea of the book seems to be to furnish the student the tools and material, and by the aid of helpful questions where needed, to have him work up his ideas for himself, in all cases leaving some actual work and thought to the student himself. As distinguishing features we notice: A constant effort to keep prominent the connection between geometrical relations and their applications in the arts; the early introduction and use of the notions of locus and of symmetry; distinction between the obverse and reverse of plane figures; and the closeness of relation between regular chains, polygons, and the circle. There are numerous exercises and problems. It must be left to actual trial to determine its adaptation to class use. J. M. C.

The Alumni Bulletin of the University of Virginia, for November, contains an appreciative sketch, with portrait, of our esteemed subscriber, Professor Charles Scott Venable, LL. D., who lately retired from the head professorship of mathematics at the University of Virginia, a position he has held for over thirty years. J. M. C.

We have received the following valuable papers, in pamphlet form, from Dr. Artemas Martin, editor of the *Mathematical Magazine*: "About Cube Numbers whose Sum is a Cube Number"; About Biquadrate Numbers whose Sum is a Biquadrate Number"; Notes about Square Numbers whose Sum is either a Square or the Sum of other Squares"; On Fifth-Power Numbers whose Sum is a Fifth Power"; and "Solutions of the 'Duck' Problem." Those interested in the subjects of which these papers treat cannot afford to miss them.

The last number of the *Magazine*, issued in May, 1896, contains the paper on Biquadrate Numbers, and the second installment of that on Cube Numbers. Three interesting problems are solved and ten new ones are proposed. J. M. C.

The following periodicals have been received : Journal de Mathématiques Elémentaires, (1er December, 1896) ; American Journal of Mathematics, (October, 1896) ; The Mathematical Gazette, (October, 1896) ; L' Intermédiare des Mathématiciens, (November, 1896) ; Miscellaneous Notes and Queries, (December) ; The Kansas University Quarterly, (October, 1896) , The Monist, (October, 1896) ; Bulletin of the American Mathematical Society, (December, 1896) ; The Educational Times, (November, 1896); The Mathematical Review, (July, 1896) ; The Mathematical Magazine, (No. 10, issued in May, 1896) ; Annals of Mathematics, (September, 1896). J. M. C.

ERRATA.

On page 221, for "numbers" read *terms* in line 16.

In solution of problem 42, page 220, the part under Example 2, reading, "For $p-q$, $a=9/2$, $b=13/2$, etc.," should be under Example 1, to tally with "for $p+q$, etc."

Page 234, line 5, for "$\rho^{-\frac{1}{3}}$" read $\rho^{-\frac{1}{2}}$.

Page 234, line 5, for $\sqrt{\rho+\rho}$" read $\rho+\sqrt{\rho}$.

Page 243, line 5, omit decimal point in denominator.

Page 258, in Figure, read D for "B" and B for "D".

Page 259, multiply the numerator of the right hand member in the value of p by 2.

Page 288, problem 38, the figure is wrong. The arc CE should be *parallel* to BA, as the solution says. Also, BC, which is an arc of the horizon, should be in a level plane.

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